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3 DETERMINING THE OPTIMUM SIZE OF

RENTED AND OWNED FLEETS 1/

by

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ABSTRACT

The problem treated is: Find the optimum pool size where a pool of vehicles serves the needs of a group of people and equipment may be rented and/or owned.

The planning period under consideration is 1 year, which is divided into periods. Demands come from separate distributions for each period. Three distinct ways of supplying equipment are of interest:

1. Owning all equipment.
2. Renting all equipment.
3. Both owning and renting equipment.

The number of owned vehicles is held constant throughout the year, but the number of rented vehicles might vary from period to period. Once an optimum schedule is determined, changes are not allowed until the end of the year at which time a new optimum schedule may be determined. Each pool of equipment is treated as a multichannel queueing system with parallel channels--a common waiting line. Demands for equipment are treated as arrivals to the queue and waiting time results in shortage costs. Waiting time is treated as a function of the number of channels.

Suppose the optimum number of owned vehicles for some year is smaller than the optimum for the preceding year. It is reasonable to assume that the difference will not be large and can be taken care of by omitting to replace some of the vehicles already scheduled for replacement. Accordingly we begin the 1-year planning period at the time of replacement and assume vehicles are not replaced at any other time. Finally, we assume setup costs are incurred only at the time vehicles are rented and that both rental and ownership costs are proportional to the number of vehicles for each period.

INTRODUCTION

Measurement of Equipment Needs

Precise tools are needed to determine whether organizations have too much equipment or too little. Benefits from employing equipment should be balanced by the costs of providing the equipment; too much equipment results in excessive ownership, rental, and setup costs, whereas too little equipment results in excessive shortage costs. A method is needed for determining the best balance.

The traditional ways of determining how much equipment is needed are:

--Arbitrary assignment by published criteria (e.g., Assignment standards in handbook)

--Standards of utilization (minimum period of usage or minimum mileage)

--Judgment of management (intuition)

The assignment of vehicles to an individual, for his exclusive or primary use is also often traditional.

In this paper we shall discuss a way of determining how much fleet equipment should be owned and/or rented.

Some of the traditional methods of measuring utilization are as follows:

1. Express the hours of use as a percentage of total hours available.
2. Count those days which show any sort of use and express the count as a percentage of total available days.

Both of these methods have shortcomings because they portray only "service;" i.e., they show only when a piece of equipment was busy.

They do not show whether some people had to postpone trips because the equipment was busy.

Focus on Fleet-Type Analysis

In this study we focus our attention on equipment which can serve the needs of more than one person or functional group. We call such a group a pool. The U. S. Forest Service operates large numbers of such pools throughout the United States. These pools will provide the primary examples for this paper. Some examples of Forest Service pools are: firefighting aircraft within appropriate regions; sedans and pick-up trucks for those headquarters which are big enough to warrant pooling of equipment, as pooling is defined. The reason for including two equipment types is that the techniques of analysis are similar for studying both a fleet of aircraft and a fleet of vehicles. What is learned from studying one fleet will be useful in studying the other.

Statement of Problems

The tools of analysis we prepose to develop are to be used on the following type of problems for an annual planning period:

A. Optimum pool size--all equipment owned.--Assuming the supply of equipment for each operating pool consists only of owned equipment, what is the optimum number of pieces of equipment for each pool?

B. Optimum pool size--all equipment rented.[1/]--When demand varies seasonally, if the number of pieces rented is allowed to vary during the year, what is the optimum policy for scheduling rentals?

[1/] Included within the definition of rented vehicles are those old vehicles designated for replacement which are held over during peak periods even though replacements have already been received.

The Queueing System

We begin by dividing the year into a finite number of periods. Assume the parameters come from separate distributions for each period. These distributions are not necessarily similar. We have chosen, for simplicity, a sample of a steady state, Poisson Arrival-Exponential Service System.

We obtain shortage costs by considering each pool as a multi-channel (parallel) queueing system with a common queue, each vehicle corresponding to a single channel. Demands for equipment are treated as arrivals to the system. Each trip corresponds to a service completion.

For simplicity, we assume the vehicles (channels) have the same mean service rate, and that queue discipline is first come first served. There are two distinct but equivalent ways of arguing the queueing process--the "limited-length queue" and the "servicing of machines." Both lead to the same set of steady-state equations. The former involves the idea of a "truncated" arrival process, while the latter does not and therefore is perhaps easier to relate to the actual physical situation. Feller (1957) describes a model of "m" machines serviced by "r" repairmen which is analogous to our queueing process: we label the machines as customers, the servicemen as vehicles, and a machine breakdown calling for service becomes a customer demand for a vehicle. Feller gives the detailed arguements starting with first principles which we will not repeat; we merely restate the steady-state equations as a beginning of the detailed queueing discussion.

From analysis of the queueing system, we will obtain equations for the expected value of the wait in queue for each period. The mathematical model uses a general expression $\bar{W}_{qi}(X_i + Y)$ for the mean wait in queue for the ith period without regard for its explicit form; therefore, the probability laws governing arrivals and services need not be the same for each of the "m" periods.

Description of Costs

Vehicles must be modified to be suitable for Forest Service use. Ordinary vehicles are not suited to emergency use in firefighting and for operating on unpaved roads. Such modifications account for the bulk of setup cost "A" (dollars per unit) for rented equipment; however, the owned equipment setup cost is amortized over the lifespan and becomes part of "B" the overall ownership cost per unit. Accordingly, we assume setup cost is incurred only for rentals and only when they are first acquired, not when they are returned to the rental agency.

Ownership cost "B" (dollars per unit) is computed as follows: The total costs over the life of a vehicle including initial investment, setup, interest on investment, operations, and maintenance are divided by the total number of periods in its life to arrive at a series of equal payments amounting to "B" per period. The purpose of this over-simplification is to put rental and ownership costs on the same basis. By so doing, we overestimate the variable costs during slack periods and underestimate them during busy periods. Variable costs which are approximately the same for rented and owned equipment should be omitted from calculations of cost coefficients "B" and "d_i." This value may be recomputed each year to take account of changes in the size of the owned fleet as well as changes in the cost items mentioned above.

Rental cost "$\underline{d}$" per unit includes rental agency fees and costs of operating and maintaining the equipment. It is proportional to the number of vehicles rented per period.

Shortage cost (dollars per unit time) "$\underline{K}$" converts mean waiting time $\bar{W}(M)$ into shortage cost. Waiting time means time spent by users who postpone their trips while equipment is busy. The most serious difficulty in matching actual characteristics of real-life operations is in estimating shortage costs. How does one answer the question: What is the cost of an hour's wait? Could the waiting customer work at something else meanwhile or was the time completely lost? If the trip was for an emergency (such as firefighting), the cost of waiting is much higher. One would expect that the cost of waiting varies seasonally, being higher during the busy summer season. Passenger-carrying vehicles are employed primarily to save people's time; thus, we might relate shortage cost to the wages of those who are waiting.

From year to year, the characteristic "mix" of the fleet may change, necessitating an adjustment of the cost coefficients.

MATHEMATICAL FORMULATION

The Objective Function

For the ith period of the overall annual planning period, ($i = 1, 2, \ldots, m$) let:

d_i = Rental cost per vehicle during period "i."

X_i = Number of vehicles rented during period "i."

Y = Number of vehicles owned during the year (constant over all periods).

M_i = Number of vehicles available (both rented and owned) during period "i."

$M_i = X_i + Y$.

$\bar{W}_{qi}(M_i)$ = Expected waiting time during period "i" (when M_i Channels are available).

K_i = Cost per unit time of wait during period i.

For each of the "m" periods let:

A = Setup cost per rented vehicle.

B = Ownership cost per owned vehicle per period.

Then for the ith period:

$d_i X_i$ = Rental cost for renting X_i vehicles.

BY = Ownership cost for owning Y vehicles.

$K_i \bar{W}_{qi}(M_i)$ = Shortage cost for having M_i vehicles available ($M_i = X_i + Y$).

$A \max(0, X_i - X_{i-1})$ = Setup cost for renting additional vehicles

X_0 = Initial level of the rented fleet at time zero.

Total cost for "ith" period is:

$$A \max(0, X_i - X_{i-1}) + d_i X_i + K_i \bar{W}_{qi}(M_i) + BY \qquad (1)$$

For "m" periods we have total cost:

$$T_m = \sum_{i=1}^{m} [A \max(0, X_i - X_{i-1}) + d_i X_i + K_i \bar{W}_{qi}(M_i)] + BY \qquad (2)$$

For the case of only rented equipment, we set $Y = 0$ and $M_i = X_i$ giving:

$$T_m(\text{rent}) = \sum_{i=1}^{m} [A \max(0, X_i - X_{i-1}) + d_i X_i + K_i \bar{W}_{qi}(M_i)] \qquad (3)$$

For the case of only owned equipment we set all $X_i = 0$ and $M_i = Y$ giving:

$$T_m(own) = \sum_{i=1}^{m} [K_i \bar{W}_{qi}(Y)] + mBY \qquad (4)$$

Mean Wait in Queue

For the ith period (i = 1, 2, ..., m) let:

λ_i = Mean rate of demands per unit time.

$\frac{1}{\mu_i}$ = Mean length of time per trip (mean service time).

N_i = Maximum number of people who demand equipment.

M_i = Number of pieces of equipment (channels) available.

$\bar{W}_{qi}$ = Mean length of time waiting in line.

$\bar{L}$ = Mean number of people in the system both waiting in line and occupying vehicles.

$\bar{L}_q$ = Mean number of people waiting in line only.

p_n = Prob.(the number of people in the system is "n").

$\rho = \frac{\lambda}{\mu}$

Subscripts ("i") are now dropped since the steady-state equations are independent of the period of the year.

Assuming Poisson Arrivals and identical Exponential Services times, we have the following solutions to the steady-state equations:

$$p_n = p_0 \rho^n \binom{N}{n} \quad \cdots\cdot \quad (0 < n \leq M) \qquad (5)$$

$$p_n = p_0 \rho^n \binom{N}{n} \frac{n!}{M!\, M^{(n-M)}} \quad \cdots\cdot \quad (M \leq n \leq N) \qquad (6)$$

Let:

$\bar{L}$ = Expected value of n, the number in the entire system.

$$\bar{L} = \sum_{n=1}^{\infty} np_n$$

or:

$$\bar{L} = \sum_{n=1}^{M} n\rho^n p_0 \binom{N}{n} + \sum_{n=M+1}^{N} n\rho^n p_0 \binom{N}{n} \frac{n!\, M^M}{M!\, M^n} \qquad (7)$$

We need a convenient expression for $\bar{W}_q$ the mean wait in queue. Rather than argue from first principles, which is tedious, and because it is convenient to use later in a discussion of discrete-convexity of the objective function, we will use a shortcut, namely, that the mean wait in queue is a linear function of the mean number waiting in queue. Justification for this may be found in Little (1960). He shows that if the three means $\bar{L}$, λ, $\bar{W}$, are finite with corresponding stochastic processes strictly stationary, and, if the arrival process is metrically transitive with nonzero mean, then $\bar{L} = \lambda\bar{W}$.

Where:

$\bar{L}$ = Mean number in the system.

$\bar{W}$ = Mean time spent by a unit in the system.

$\frac{1}{\lambda}$ = Mean time between two consecutive arrivals to the system.

Following Little (1960), we define the system as the queue itself and let $\bar{L}_q$ and $\bar{W}_q$ refer to the mean number and wait in queue and obtain: $\bar{L}_q = \lambda\bar{W}_q$. Our queueing system has finite means, is strictly stationary, and, as Doob (1953 p. 460) points out, is metrically transitive because the random variables representing the lengths of time between arrivals in a Poisson process are mutually independent and from a common distribution.

The delay in the system is composed of two parts, the delay in queue and the delay during service. Taking expectations and using $\bar{L}/\lambda$ as the expected wait in the system we have:

E[delay in the system] = E[delay in queue] + E[delay in service]

or:

$$\frac{\bar{L}}{\lambda} = \bar{W}_q + \frac{1}{\mu} \tag{8}$$

Using equations (7) and (8), we arrive at a convenient, closed-form expression for the mean wait in queue:

$$\bar{W}_q = \frac{-1}{\mu} + \frac{p_0}{\lambda}\left\{\sum_{n=1}^{M} n\rho^n \binom{N}{n} + \sum_{n=M+1}^{N} n\rho^n \binom{N}{n} \frac{n!\, M^M}{M!\, M^n}\right\} \tag{9}$$

It may happen that the values for mean wait in queue obtained from the finite queue model may be closely approximated by the infinite queue model. Since the infinite queue model equations are more tractable, it might be used instead. Therefore, we list the corresponding equation with a "o" superscript to denote that infinite queue is allowed.

$$\bar{W}_q^o = \frac{-1}{\mu} + \frac{p_0\, \rho^M}{M!\, M\mu\left(1-\frac{\lambda}{M\mu}\right)^2} \tag{10}$$

Maximum queue length equals $(N_i - M_i)$, the difference between N_i the maximum number of people who demand equipment and M_i, the number of channels (vehicles). When the number of vehicles M_i is large enough to allow one for each user, i.e., $M_i = N_i$, then no queue can exist and mean wait is zero (which suggests why we assume a finite-length queue model). N_i may be larger for some periods than others and an optimum value for Y may be larger than some N_i, smaller than others. However, an optimum value for X_i may not exceed N_i for to do so is to have a surplus of vehicles. Thus, the rented equipment (X_i) may vary in number but is bounded as follows: $X_i \leq N_i$ $(i = 1, 2, \ldots, m)$.

"Discrete-Convexity"

Later we shall investigate algorithms for finding minimums for equations (5), (6), and (7) in which we shall use the property that our objective functions are "discrete-convex;" i.e., it has non-negative, second-order, forward finite differences.

We investigate the property of discrete-convexity for both finite and infinite queue models in the appendix, in detail. For the moment, we are concerned only with the property that for both models, the mean wait in queue is a discrete-convex function of the number of channels, i.e., $\bar{W}_{qi}(M_i) = \bar{W}_{qi}(X_i + Y)$ is discrete-convex.

Next, we examine the individual terms of the objective function and show that each is discrete-convex. Both C_iX_i and mBY are linear and therefore discrete-convex. That $\max(0, X_i - X_{i-1})$ is discrete-convex is not so obvious that a graph would not help to see this:

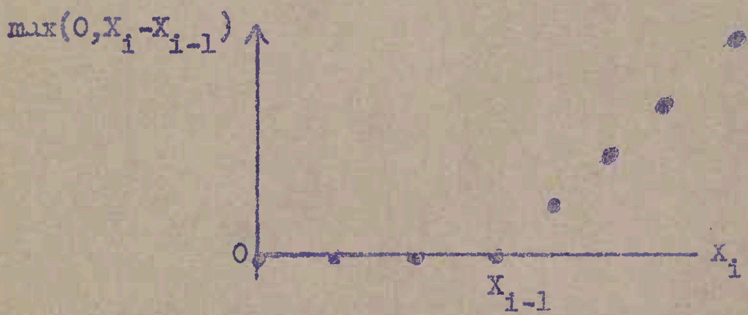


Figure 1

We could join discrete points by straight lines to form a continous convex function from each of our integer-valued functions. Since the positive sum of convex functions is a convex function, we have shown that if our objective function "T" were made into a continuous function, it would be a convex function (with the desired property of having non-negative, second-order, forward finite differences). Finally, restricting X_i and Y to the integers doesn't change this non-negativity and we have shown that our objective functions are discrete-convex.

Global and Local Extrema

In the inventory problems treated in the literature, the quantity of a commodity is usually treated as a continuous variable; hence, for convex objective functions, it follows that a local minimum is also a global minimum. This is also true for the multiple-period model function of several variables. However, in our case, expected wait in queue is defined only on integers. Elsewhere, we showed our objective function to be "discrete-convex" (in the sense that second-order, forward finite differences with respect to each variable are non-negative). But, a discrete-convex function of several variables can have a local minimum which is not a global minimum. We will show an example in the appendix.

Furthermore, we have an additional complication due to the discreteness of the variables. Suppose we could redefine our objective function as one of continuous variables with the same functional values whenever the variables are integers, (i.e., define a smooth surface through the vector points in m-space). Then, assuming a convex function, we still cannot guarantee that having found its absolute minimum, we can round off the continuous variables to integers and still have the solution to the corresponding problem which was restricted to the integers.

The example in the appendix illustrates this idea as well as the property that a global minimum need not be coincident with a local minimum, for a discrete-convex function. For this example, the author is indebted to his colleague Sherman J. O'Neill.

Further analysis is necessary to check if the equation for our objective function exhibits the "trough-like" characteristic of this example and if so, whether in any real case it might not give a good approximation to the integer-valued problem.

For both convex and discrete-convex functions, the minimum may lie on a boundary of the region on which the functions are defined rather than at an extremum. Any solution procedure must take this into account.

However, the situation is not hopeless. For the case of owning all equipment, we have only one variable, "Y" and, for discrete-convex functions of one variable, the foregoing difficulties don't exist because a local minimum is also a global minimum. For the cases of (a) rent only, and (b) rent and/or own, we shall introduce dynamic programming to overcome the foregoing difficulties, checking boundary points as well as achieving an absolute minimum.

ALGORITHMS

Dynamic Programming Solution of: Rent and/or Own Problem

Rather than discuss a separate algorithm for the case of rent only, we shall discuss the more general situation of rent and/or buy, treating rent only is a special case when Y = 0. Using the recursion methods of dynamic programming seems appropriate, because the variables X_i, M_i, and n are integers (See the preceding discussion on convex and discrete-convex functions of several variables.) Furthermore these recursion methods lead to an absolute minimum (Bellman and Dreyfus, 1962).

The key to a dynamic programming formulation of our problem lies in developing a recursion relationship between two adjacent periods, "i" and "i-1." This is called a "functional equation." We begin with a one-period model, then with a two-period model, and generalize the results to an m-period model.

One-Period Model

As before, Y is the number of owned vehicles. For the moment, consider Y as a fixed number. Since the maximum number of vehicles we could supply is bounded by the maximum number of people the vehicles serve, we have the restrictions:

$$0 \leq Y \leq N_{max}$$

$$0 \leq X_i \leq N_i \leq N_{max} \quad (i = 1, 2, \ldots, m)$$

where: N_{max} = maximum value any N_i may take on for any of the "m" periods, i.e., $N_{max} = \max_i \{N_i\}$ $(i = 1, 2, \ldots, m)$. Let: X_0 = initial size of rented fleet at time zero. $(X_0 = 0, 1, 2, \ldots, N_{max})$. The total cost for a one-period model is:

$$T(X_1, X_0, Y) = d_1 X_1 + A \max(0, X_1 - X_0) + K_1 \bar{W}_{qi}(X_1 + Y) + BY \tag{11}$$

We have designated T as a function of X_0 and Y as well as X_1 because, although we treat X_0 and Y as fixed for now, later we will let them vary and the dependence of our answer on X_0 and Y should be kept in mind. Minimizing $T(X_1, X_0, Y)$ with respect to X_1 yields a new expression $C_1(X_0, Y)$ which we designate as a function of X_0 and Y with the understanding that for the time being X_1 is the only variable:

$$C_1(X_0, Y) = \min_{\{X_1\}} [T(X_1, X_0, Y)] \tag{12}$$

where:

$$\{X_1\} = \{X_1 \mid 0 \leq X_1 \leq N_1 - Y\}$$

Thus:

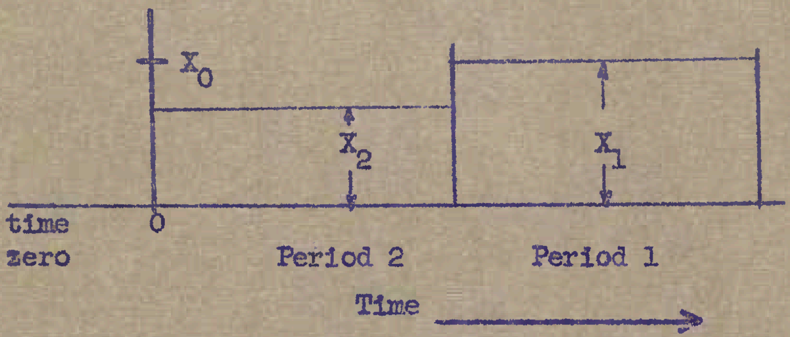
$$C_1(X_0, Y) = \min_{\{X_1\}} [d_1 X_1 + A \max(0, X_1 - X_0) + K_1 \bar{W}_{q1}(X_1 + Y) + BY] \tag{13}$$

Setting $Y = 0$ gives us the expression for the case of rent only, namely:

$$C_1(X_0,0) = \min_{\{X_1\}}[d_1X_1 + A\max(0, X_1-X_0) + K_1\bar{W}_{q1}(X_1)] \tag{14}$$

M-Period Model

Consider the case of a two-period model. Renumber the periods, such that the first period in time follows the second period along the time axis. A graph of a sample two-period situation may help to see the relationships:



Period 2 begins with an initial rented fleet size of X_0; however, the initial rented fleet size for period 1 is X_2. Beginning at time zero with X_0, again holding Y fixed and rent $X_2 = \hat{X}_2$ vehicles for period 2, we want to minimize the cost for the remaining period 1, which by definition is $C_1(\hat{X}_2,Y)$; so our cost for renting $X_2 = \hat{X}_2$ would be defined as:

$$T_2(\hat{X}_2,X_0,Y) = d_2X_2 + A\max(0,\hat{X}_2-X_0) + K_2\bar{W}_{q2}(\hat{X}_2+Y) + C_1(\hat{X}_2,Y_0) \tag{15}$$

for a two-period process.

An optimal choice for X_2 is one which minimizes this function; accordingly, letting X_2 again become a variable we can define:

$$C_2(X_0,Y) = \min_{\{X_2\}} T_2(X_2,X_0,Y) \tag{16}$$

$$C_2(X_0,Y) = \min_{\{X_2\}}[d_2X_2 + A\max(0,X_2-X_0) + K_2\bar{W}_{q2}(X_2+Y) + BY + C_1(X_2,Y)] \tag{17}$$

where: $$\{X_2\} = \{X_2 \quad 0 \leq X_2 \leq N_2 - Y\}$$

It follows immediately from the preceding two-period discussion that the general functional equation for an m-period model is:

$$C_m(X_0,Y) = \min_{\{X_m\}}[d_mX_m + A\max(0,X_m-X_0) + K_m\bar{W}_{qm}(X_m+Y) + BY + C_{m-1}(X_m,Y)] \tag{18}$$

where: $$\{X_i\} = \{X_i \mid 0 \leq X_i \leq N_i - Y\} \tag{19}$$

$$(i = 1, 2, \ldots, m)$$

Rent Only

As before, setting $Y = 0$ we have a <u>Multiperiod Rent Only Model</u>, namely:

$$C_m(X_0,0) = \min_{\{X_m\}}[d_mX_m + A\max(0,X_m-X_0) + K_m\bar{W}_{qm}(X_m) + C_{m-1}(X_m,0)] \tag{20}$$

For a fixed Y, we have shown how to find $C_m(X_0,Y)$. Repeating the procedure for each value assigned to Y in which $(0 \leq Y \leq N_{max})$, $(N_{max} = \max_i\{N_i\})$.

That $C_m(X_0,Y)$ is discrete-convex is not obvious. To show this, we argue inductively starting with

$$C_1(X,Y) = \min_{\{X_1\}} T(X,X_1,Y)$$

$$C_1(X,Y) = \min_{\{X_1\}}[d_1X_1 + A\max(0,X_1-X) + K_1\bar{W}_{q1}(X_1+Y) + BY] \qquad (13)$$

We have already discussed the discrete-convexity of $T(X,X_1,Y)$ for any given value of X_1. Now, we show that T is discrete-convex in X for fixed (X_1,Y), and is discrete-convex in Y for fixed (X_1,X).

For a given X_1, $\max(0,X_1-X) = \begin{Bmatrix} X_1-X \text{ for } X_1 \geq X \\ 0 \text{ otherwise} \end{Bmatrix}$ and is graphed below:

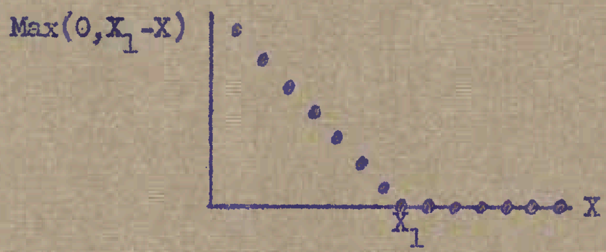


Figure 2

If we joined the discrete points by straight lines, the result would be a convex function with non-negative, second-order, forward finite differences; and restricting that convex function to the integers doesn't affect that non-negativity. Hence, $\max(0,X_1-X)$ is discrete-convex. We have already discussed the discrete-convexity of $\bar{W}_{q1}(M_1)$ where $M_1 = X_1+Y$, so that for any fixed X_1, $\bar{W}_{q1}(Y)$ is discrete-convex. "BY" is linear and thus discrete-convex. Finally, treating each of the terms in (13) as a function of two variables, X_1 and Y, so that they are all defined on the same domain, we may add together discrete-convex functions to obtain another discrete-convex function, namely, $C_1(X_0,Y)$.

We turn now to examine the discrete-convexity of

$$C_2(X_0,Y) = \min_{\{X_2\}} T(X_2,X_1Y) \tag{16}$$

$$= \min_{\{X_2\}}[d_2X_2 + A\max(0,X_2-X) + K_2\bar{W}_{q2}(X_2 + Y)$$

$$+ BY + C_1(X_2,Y) \tag{17}$$

where $\{X_2\} = \{X_2 \mid 0 \leq X_2 \leq N_2-Y\}$.

Examining the last term of the above functional equation, for fixed X_2,Y,

$$C_1(X_2,Y) = \min_{\{X_1\}}[d_1X_1 + A\cdot\max(0,X_1-X_2) + K_1\bar{W}_{q1}(X_1 + Y) + BY$$

which is an equation of the same form as $C_1(X_0,Y)$ of which the discrete-convexity arguements have already been given. The remaining terms are also of the same form as for $C_1(X_0,Y)$, so the previous arguements again go through.

Finally, assuming that $C_{m-1}(X_0,Y)$ is discrete-convex we show that $C_m(X_0,Y)$ is also discrete-convex by again referring to the previous arguements for the case of $C_1(X_0,Y)$.

Computation Procedure

1. Let X be the number of rented vehicles at the beginning of the highest numbered period. Compute for each value of $X = 0, 1, 2, \ldots N_{max}$, each value of $X_1 = 0, 1, 2, \ldots, N$, and $Y = Y^{(i)}$ a table of:

$$V_1(X,X_1,Y^{(i)}) = [d_1X_1 + K_1\bar{W}_{q1}(X_1 + Y^{(i)}) + BY^{(i)} + A\max(0,X_1-X)].$$

From this table, select $\min V_1(X,X_1,Y^{(i)}) = C_1(X,Y^{(i)})$ for each value of X giving a table of $\{C_1(X,Y^{(i)})\}$ values from which we may subsequently draw to evaluate $C_1(X_2,Y^{(i)})$.

2. Compute for each value $X = 0, 1, 2, \ldots, X_{max}$, each value of $X_2 = 0, 1, 2, \ldots, N_2$ and again fix $Y = Y^{(1)}$, a table of:

$$V_2(X, X_2, Y^{(1)}) = [d_2X_2 + K_2\bar{W}_{q2}(X_2 + Y^{(1)}) + BY^{(1)} + A \max(0, X_2 - X)].$$

For each value of X_2 we add $C_1(X_2, Y^{(1)})$ from step 1 giving another table of:

$$C_1(X_2, Y^{(1)}) + V_2(X, X_2, Y^{(1)})$$

for all values of X from which for each X we can select the minimum value giving a table of $C_2(X, Y^{(1)})$ from which we may subsequently draw to evaluate $C_2(X_3, Y^{(1)})$.

Simiarly we continue until we have finally evaluated $C_m(X_0, Y^{(1)})$. Throughout the preceding discussion, we kept Y fixed at $Y = Y^{(1)}$. To complete the solution procedure we repeat the foregoing for all values of Y over its range, $(0 \leq Y \leq N_{max})$ in which $N_{max} = \max\{N_i\}$. For each value Y assumes, compute the optimal values $X_1(Y), X_2(Y), \ldots, X_m(Y)$ together with the associated $C_m(X_0, Y)$ and from such a table we can select the final answer. Rather than store the results for each value chosen for Y, we can: compute $C_m(X_0, Y^{(1)})$ and $C_m(X_0, Y^{(2)})$ for $Y = Y^{(1)}$ and $Y = Y^{(2)}$ repsectively; compare the two, discard the larger, store the smaller; compute $C_m(X_0, Y^{(3)})$ for $Y = Y^{(3)}$ compare this with the previously stored value, discard the smaller, etc.

So far we have only briefly discussed how to evaluate $C_i(X_0, Y)$ for any given $i = 1, 2, \ldots, m$ and any given Y. Since $C_i(X_0, Y)$ is integer-valued, methods of calculus, & convex programming are not applicable. A thorough discussion of this problem lies beyond the scope of this paper. (An optimal search technique, using Fibonacci Numbers for locating the coordinates of a maximum for unimodal functions

is discussed in Bellman and Dreyfus (1962, p. 152)). However, for problems of the size we expect to encounter (12 periods, $\max_i \{N_i\} \leq 40$), the following more simple-minded method seems adequate. For example, in $C_1(X_0,Y) = \min_{X_1}[d_1 X_1 + A \max(0, X_1 - X_0) + K_1 \bar{W}_{q1}(X_1 + Y) + BY]$ we have a simple linear relationship in each term except for shortage cost $K_1 \bar{W}_{q1}(X_1 + Y)$, which is a function of $\bar{W}_{q1}(X_1 + Y)$ the mean wait in queue. For convenience, we repeat this latter expression:

$$\bar{W}_{q1}(X_1 + Y) = \frac{-1}{\mu} + \frac{p_0}{\lambda}\left\{\sum_{n=1}^{X_1+Y} n\rho^n \frac{N_1}{n} + \sum_{n=X_1+Y+1}^{N_1} n\rho^n \binom{N}{n} \frac{n!\, M_1^{M_1}}{M_1!\, M_1^n}\right. \tag{9}$$

This is a closed form and can be computed beforehand for appropriate values of $X_1 + Y$, keeping in mind the range: $0 \leq (X_1+Y) \leq \max_i \{N_i\}$, $(i = 1, 2, \ldots, m)$. The remaining terms $d_1 X_1$, BY, $A \max(0, X_1 - X_0)$ are easy to compute and store so that finding a minimum is straightforward.

Referring to equation (2), the original objective function of $m + 1$ variables has now been reduced to m discrete-convex equations of one variable (for each value of $Y = 0, 1, 2, \ldots, N_{max}$). Hence, at each stage the minimizing procedure guarantees an absolute minimum may be found.

Alternate Algorithm for Own All Equipment Problem

In this case since no equipment is rented we set all $X_i = 0$ our problem becomes: minimize

$$T_m(Y) = mBY + \sum_{i=1}^{m} K_i \bar{W}_{qi}(Y) \tag{21}$$

where Y is an integer. Defining Y* as the value of Y which minimizes T, and since $T_m(Y)$ is discrete-convex, we have the following conditions which Y* must satisfy:

$$\Delta T(Y^*) < 0 \qquad (22)$$

$$\Delta T(Y^*+1) \geq 0$$

where $\Delta T(Y^*)$ is the first forward finite difference at Y*.

Finding Y* satisfying (22) involves a search technique of which there are numerous varieties. Since T(Y) is discrete-convex, this property can probably be employed to find a more optimal search technique than the one following. For problems of the size we expect to encounter, namely where $Y < 100$, what follows appears to be a workable method.

To find the Y* which satisfies these equations, assume Y* is unique, then begin with Y = 1, and continue increasing Y by one. At each step, solve for the finite differences indicated by equations (22). If at the onset $\Delta T(Y = 1) \geq 0$ then optimality has already been reached and no further steps are necessary.

The procedure outlined above will produce a global minimum at Y* since for discrete-convex functions of one variable, a local minimum has the same functional value as the global minimum. If Y* is not unique, equations (22) would read:

$$\Delta T(Y^*) < 0 \qquad (23)$$

$$\Delta T(Y^*+1) = 0$$

which indicates that the choice between Y* and (Y*+1) is optional from a purely economic viewpoint. Hence, a decision is based on other qualitative grounds. Accordingly, we have shown how to find the optimum pool size when all equipment is owned.

CONCLUSIONS

Referring to the cost of wait, for the Forest Service, these "values of postponement" can be collected as part of the overall data collection scheme and could constitute a wide survey which in itself would be of value to the management of any Region wishing to analyse vehicle usage of its National Forests or Ranger Districts.

Following are some possibilities for obtaining costs of waiting (postponement). We employ aircraft primarily as a firefighting weapon and a shortage of aircraft results in increased fire spread; thus, we might relate shortage cost to the effectiveness of aircraft in fighting fires and the cost per unit time of fire spread.

If all methods of estimating shortage cost fail to produce reliable, absolute values, we can still use these estimates in studying renting of equipment compared with owning it because the shortage costs are the same for both. We can also let waiting cost per unit time be a variable and study the sensitivity of the model to changes in waiting costs.

Referring to "B" and "d_i", the ownership and rental costs, a more accurate approximation would result in separating each of these costs into two parts, fixed costs and variable costs. Then, variable costs could be charged on the mean number of vehicles in use as well as the mean service time (trip length). However, for this study we use a fixed coefficient "B" as a first approximation.

Referring to the assumption of steady-state, most motor pools familar to the author are characterized by having trips of short duration, with a few lasting several days. If one lays the results of each day's operations end to end, we might approximate a realization of a renewal process, an example of which is the Poisson process.

Although the dynamic programming functional equations are formulated without regard for properties of the objective function, we conjecture that some more efficient means may be devised to perform the minimizing operation at each stage of the calculations. The reason is because the objective function as well as $C_m(X_0,Y)$ are discrete-convex and at each stage the minimizing operation is performed on a one variable problem.

APPENDIX

Proof of Discrete-Convexity Properties of the Mean Wait in Queue

Theorem: For the Poisson-exponential, finite-length queue system, the mean wait in queue is a discrete-convex function of the number of channels available during any period.

We first show the following:

Lemma: Both the mean number in the system and the mean number in the queue are discrete convex functions of the number of channels.

Proof: The expression for $\bar{L}$, the mean number in the system, is repeated for convenience.

$$\bar{L}_i = \sum_{n=1}^{M_i} n p_0 \rho^n \binom{N_i}{n} + \sum_{M_i+1}^{N_i} n p_0 \rho^n \binom{N_i}{n} \frac{n!\ M_i^{M_i}}{M_i!\ M_i^n} \tag{24}$$

For convenience in what follows we drop the subscripts which designate the period. The second-order difference is denoted by

$$\begin{aligned}\Delta^2\bar{L}(M) = {} & p_0\left\{\sum_{n=1}^{M+2} n\rho^n \binom{N}{n} + \sum_{n=M+3}^{N} n\rho^n \binom{N}{n} \frac{n!\ (M+2)^{M+2}}{(M+2)!\ (M+2)^n}\right\} \\ & - 2p_0\left\{\sum_{n=1}^{M+1} n\rho^n \binom{N}{n} + \sum_{n=M+2}^{N} n\rho^n \binom{N}{n} \frac{n!\ (M+1)^{M+1}}{(M+1)!\ (M+1)^n}\right\} \\ & + p_0\left\{\sum_{n=1}^{M} n\rho^n \binom{N}{n} + \sum_{n=M+2}^{N} n\rho^n \binom{N}{n} \frac{n!\ M^M}{M!\ M^n}\right\}\end{aligned} \tag{25}$$

This expression can be simplified by letting

$$f(n) = n\rho^n \binom{N}{n} p_0 \tag{26}$$

and noting that

$$\sum_{n=1}^{M+2} f(n) - 2\sum_{n=0}^{M+1} f(n) + \sum_{n=0}^{M} f(n) = f(M+2) - f(M+1)$$

so that $\Delta^2\bar{L}(M)$ can be written:

$$\Delta^2\bar{L}(M) = f(M+1)\left[1-2+\frac{M+1}{M}\right] + f(M+2)\left[1-2\left(\frac{M+2}{M+1}\right) + \frac{(M+2)(M+1)}{M}\right] \tag{27}$$

$$+\sum_{i=3}^{N} \frac{f(M+i)(M+i)!}{M!}\left[\frac{1}{(M+1)(M+2)^{i-1}} - \frac{2}{(M+1)^{i}} + \frac{1}{M^{i}}\right]$$

then:

$$f(M+i) \geqslant 0 \quad \text{for } (i = 1, 2, \ldots., N) \tag{28}$$

and:

$$\frac{M+1}{M} \geqslant 1 \quad \text{for } (M = 1, 2, \ldots.) \tag{29}$$

and:

$$\frac{(M+2)(M+1)}{M} > \frac{2(M+2)}{(M+1)} \quad \text{for } (M = 1, 2, \ldots.) \tag{30}$$

and:

$$\frac{1}{M^{i}} + \frac{1}{(M+1)(M+2)^{i-1}} > \frac{2}{(M+1)^{i}} \quad (i = 1, 2, \ldots.) \tag{31}$$

The second-order difference is non-negative and we have shown that $\bar{L}(M)$ is discrete-convex. Let $\bar{L}_q$ be the mean number in the waiting line.

From equation (8) we multiply by λ giving:

$$\bar{L} = \lambda\bar{W}_q + \frac{\lambda}{\mu} \tag{32}$$

or:

$$\bar{L}_q = \bar{L} - \rho \tag{33}$$

Since $\bar{L}$ is discrete-convex and since ρ is a constant, it follows that $\bar{L}_q(M)$ is discrete-convex.

Again from Litte (1961), for systems that behave like ours $\bar{W}_q = \bar{L}_q/\lambda$ so that $\bar{W}_q(M)$ is convex, since it is a linear function of $\bar{L}_q$ which establishes the theorem.

Theorem: For the "infinite-length" queue system, the mean wait in queue (in line) is a discrete-convex function of the number of channels (vehicles) available during any period.

Proof:

Let:

$\bar{W}_q^o$ = Mean wait in queue (infinite queue).

$$\bar{W}_q^o = \frac{-1}{\mu} + \frac{Q_M}{M\mu-\lambda} \tag{34}$$

Where:

$$Q_M = \frac{p_0}{M!} \; \frac{\rho^M}{M-\frac{\rho}{M}} \tag{35}$$

$$\bar{W}_q^o = \frac{-1}{\mu} + \frac{p_0\rho^M}{M!(1-\frac{\rho}{M})(M\mu-\lambda)} = -\frac{1}{\mu} + \frac{p_0\rho^n}{M!\; M\mu(1-\frac{\lambda}{M\mu})^2} \tag{36}$$

Treating this as a discrete function of M, we show that the second-order, finite differences are non-negative.

$$\Delta^2\bar{W}_q^o(M) = \bar{W}_q^o(M+2) - 2\bar{W}_q^o(M+1) + \bar{W}_q^o(M) \tag{37}$$

$$\Delta^2\bar{W}_q^o(M) = \frac{p_0\rho^M p^2}{M!(M+2)(M+1)(1-\frac{\rho}{M+2})\;[(M+2)\;\mu-\lambda]} \tag{38}$$

$$- \frac{2\,p_0\rho^M\rho^2}{M!(M+1)(1-\frac{\rho}{M+1})\;[(M+1)\;\mu-\lambda]}$$

$$+ \frac{p_0\rho^M}{M!(1-\frac{\rho}{M})\;[M\mu-\lambda]}$$

Since each of the three terms (without regard for the sign before the second term) is non-negative, it is enough to show that the second term is dominated by the third term:

$$\frac{2\rho}{(M+1)(1-\frac{\rho}{M+1})\;[(M+1)\;\mu-\lambda]} \leq \frac{1}{(1-\frac{\rho}{M})\;[M-\lambda]} \tag{39}$$

To see this, first note:

$$\frac{2\rho}{M+1} < 1 \quad (\text{For } \rho < 1,\ M \geqslant 1) \tag{40}$$

Then compare the remaining similar terms in the denominator.

First:

$$\frac{\rho}{M+1} < \frac{\rho}{M} \quad (\text{for } \rho < 1,\ M \geqslant 1) \tag{41}$$

thus:

$$1 - \frac{\rho}{M+1} > 1 - \frac{\rho}{M} \tag{42}$$

Then:

$$(M+1)\mu - \lambda > M\mu - \lambda \quad (\text{for } M \geqslant 1) \tag{43}$$

We have shown that $\bar{W}_{q1}(M_1)$, the mean wait in queue, is a discrete-convex function of $M_1 = X_1 + Y$, the number of channels.

A Discrete-Convex Function with a Local Minimum

Define a minimum of a discrete function as follows: We are at a minimum if in moving to any adjacent point in the lattice, the functional value increases or remains the same. An adjacent point means one which is reached by making a unit change in any or all of the variables. Consider the graph showing a lattice of a discrete-convex function $g(X_1,X_2)$, which is restricted to integer arguments of a convex function $f(X_1,X_2)$, defined everywhere, so that the functional value of point (X_1,Y_1) is equal to the perpendicular distance of the point (X_1,Y_1) to a diagonal line joining two nonadjacent points of the lattice.

An example is shown below; the dotted lines represent contour lines of equal functional value, $f(X_1,X_2) = 1, 2, 3, \ldots$. Only a few

contour lines of special interest have been drawn. The surface formed by $f(X_1,X_2)$ is a symmetric "trough" whose bottom $f(X_1,X_2) = 0$ lies along the diagonal line. Functional values are shown for some points indicated by square brackets on the lattice.

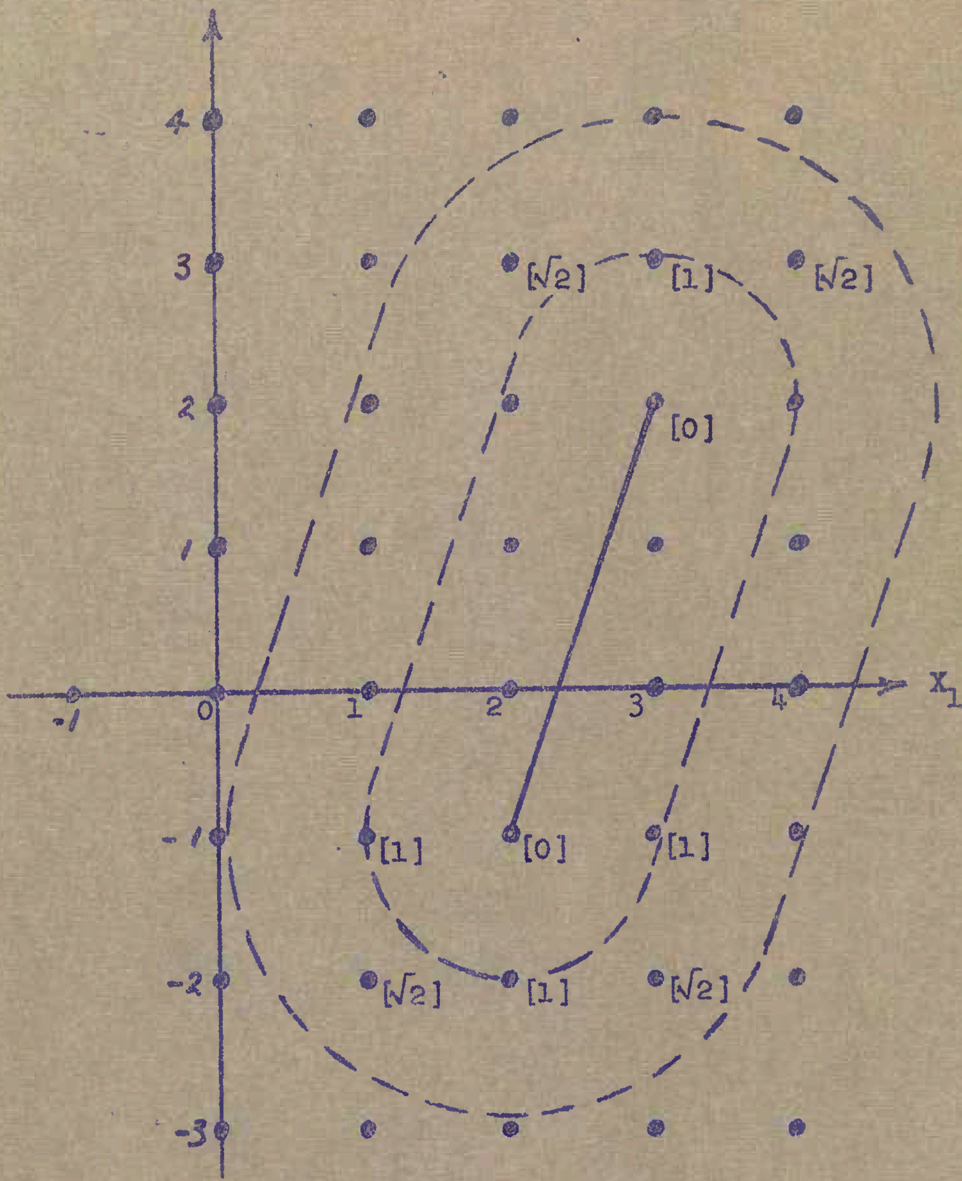


Figure 3

Now, redefine both functions by adding the plane $h(X_1,X_2) = \epsilon(3-X)$, $\epsilon > 0$. This plane intersects the X_1,X_2 plane along the line $X_1 = 3$ and slopes to the left and upwards with slope $= \epsilon > 0$. The new function, $f(X_1,X_2) + h(X_1,X_2) = j(X_1,X_2)$ for small "ϵ" is strictly convex. Restricting it to integer values of X_1,X_2 defines a new function, $k(X_1,X_2)$ which is discrete-convex. All functional values of $k(X_1,X_2)$ are slightly different from the corresponding functional values of $g(X_1,X_2)$ except for those points lying on the line $X_1 = 3$. Portions of the new functions $j(X_1,X_2)$ and $k(X_1,X_2)$ appear in Figure 4. Some functional values of $k(X_1,X_2)$ are in square brackets.

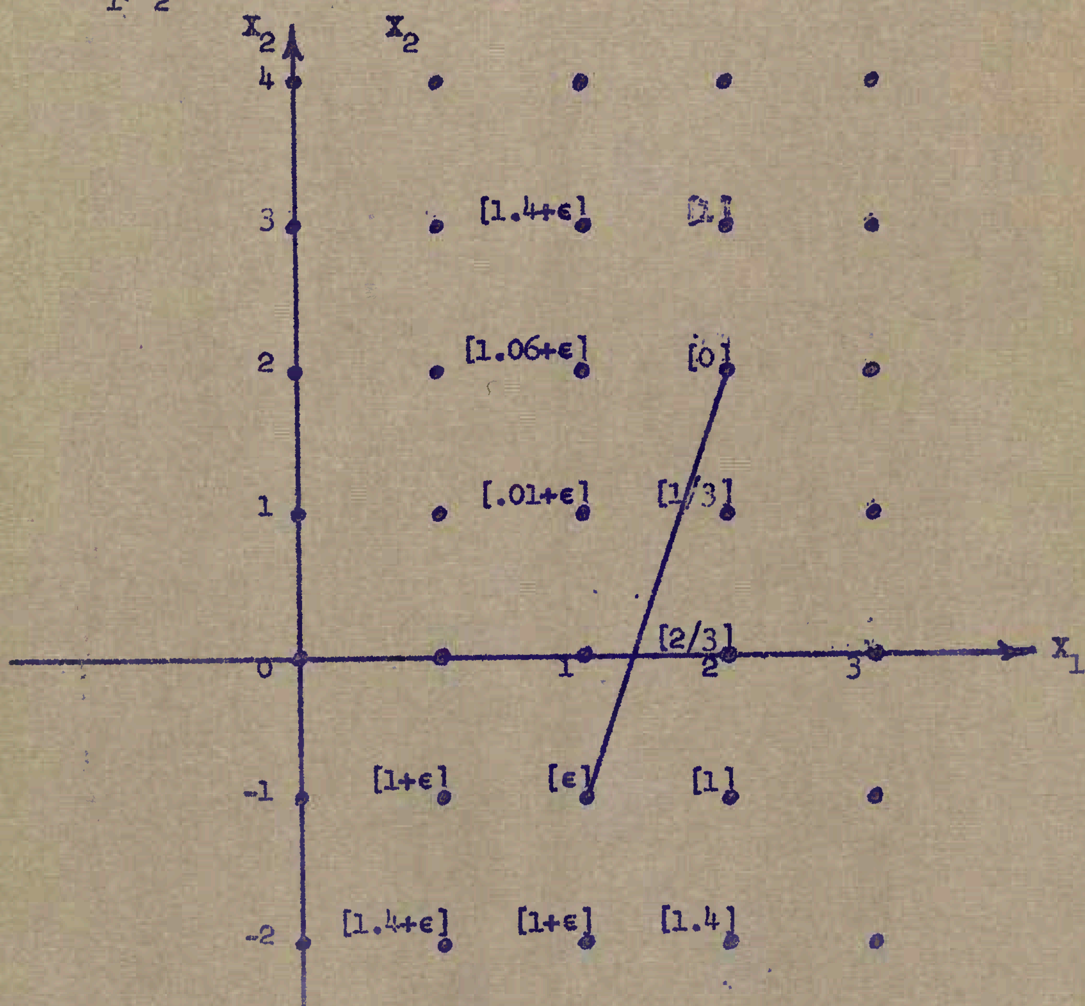


Figure 4

For suitably small values of "ϵ," the lattice point $(X_1,X_2) = (2,-1)$ has a smaller functional value than any of its adjacent lattice points, which by the definition makes $(2,-1)$ a local minimum for the integer-valued function $g(X_1,X_2)$. In contrast, the continuous function $f(X_1,X_2)$ does not have a local minimum, only an absolute minimum at $(3,2)$.

Regarding round off, consider any point on the diagonal line of $f(X_1,X_2)$ in Figure 2. The diagonal line is a nonunique, absolute minimum for $f(X_1,X_2)$; however, for many points along the diagonal, rounding off will yield an integer solution which is clearly not the solution to the integer valued problem of finding the minimum of $g(X_1,X_2)$. Only for points on the diagonal with $1 > X_2$, or $X_2 \leq 0$ will the round off yield the correct answer.

We have demonstrated that even with a convex function having found its minimum, rounding off to an adjacent integer point does not guarantee a solution to the corresponding integer-valued problem.

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